An existence theorem for fuzzy partial differential equation

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Теорема о существовании нечеткого дифференциального уравнения в частных производных

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Abstract: *in this paper, first a space of fuzzy numbers is constructed and a scalar product is introduced. The derivative of fuzzy function in this space is defined. Further, Poisson's equation with first boundary condition for fuzzy functions is considered. It is shown that if problems data (right hand site and boundary function) are fuzzy then solution of this problem also is fuzzy function.*

Аннотация: *в данной статье вначале было построено пространство нечетких чисел и представлено скалярное произведение. Была определена производная функции принадлежности нечеткого множества в данном пространстве. Далее рассматривается уравнение Пуассона с граничными условиями первого рода для функций принадлежности нечеткого множества. Показано, что если данные задачи (правая часть и граничная функция) содержит нечеткие числа, то и решение этой задачи – это функция принадлежности нечеткого множества.*

Keywords: *fuzzy partial differential equation, fuzzy function, poisson's equation.*

Ключевые слова: *нечеткие дифференциальные уравнения в частных производных, функция принадлежности нечеткого множества, уравнение Пуассона.*

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1. Introduction

The complexity of the world makes the events we face uncertain in furious forms. Besides, randomness, fuzziness and is important uncertainty, which plays an essential role in the real world. Fuzzy set theory has been developed very fast since it was introduced by scientist on cybernetics Zadeh [1] in 1965. A fuzzy set characterized with its membership function by Zadeh. For the purpose of measuring fuzzy events, Zadeh [2] presented the concept of possibility measure and the term of fuzzy variable in 1978.

To investigate fuzzy differential equations at first one has to introduce the definition of the derivative of fuzzy function. This definition must allow one to investigate ordinary and partial differential equation.

The concept of fuzzy derivative was first introduced by Chang and Zadeh [3], and it was followed up by Dubois and Prade [4], who used the extension principle in their approach.

Other methods have been discussed by Puri and Ralescu [5]. A thorough theoretical research of fuzzy Cauchy problems was given by Kaleva [6], Seikkala [7]. Fuzzy partial differential equations were formulated by Buckley [8], and T. Allahviranloo [9] used a numerical method to solve the (FPDE).

In the present work, introducing the space of fuzzy numbers, the derivative of the fuzzy function is determined. Using this approach, the method is proposed to investigate fuzzy partial differential equations.

2. The space of the pairs of fuzzy numbers

A fuzzy set A is characterized by a generalized characteristic function $\mu_A(.)$, called membership function, defined on a universe X, which assumes values in $[0,1]$. For any $\alpha \in [0,1]$ denote by

$$A^\alpha = \{x \in X : \mu_A(x) \geq a\}$$

the α - cut of A. Let $\mu_A(.)$ is an upper semicontinuous function and

$$\sup p(A) = \{x \in X : \mu_A(x) > a\}$$

is bounded set of X. A fuzzy set is a fuzzy number if $X \subset R$ and for any $\alpha \in [0,1]$, the α -cut A^α is convex and the height of A, that is, $\sup\limits_{x \in X} \mu_A(x)$ has to be equal to one. This fuzzy number usually is called convex normal fuzzy number.

Let's define by F the class of convex normal fuzzy numbers. Then for any $a \in F$ the set of α -cut of fuzzy number a the interval $a^{\alpha} = [L_a(\alpha), R_a(\alpha)]$, $\alpha \in [0,1]$, is defined ([7]). Let $a \in F$, $b \in F$ and $a^{\alpha} = [L_a(\alpha), R_a(\alpha)]$, $b^{\alpha} = [L_b(\alpha), R_b(\alpha)]$. Then α -cut of fuzzy number $a+b$ and $ka, k \ge 0$, define as $a^{\alpha} + b^{\alpha} = [L_a(\alpha) + L_b(\alpha), R_a(\alpha) + R_b(\alpha)]$ and $ka^{\alpha} = [kL_a(\alpha), kR_a(\alpha)]$, respectively.

Note that F is not a linear space (the operation of subtraction is not defined in F).

We consider the set of pairs $(a,b) \in F \times F$ and define the operation of addition, multiplication and equivalency as

$$\begin{aligned}
&(a_1,a_2) + (b_1,b_2) = (a_1 + b_1, a_2 + b_2) \\
&k \cdot (a,b) = (ka, kb), k \ge 0, \\
&(-1) \cdot (a,b) = (b,a), \\
&(a_1,a_2) = (b_1,b_2) \Leftrightarrow a_1 + b_2 = a_2 + b_1
\end{aligned} \tag{1}$$

As zero element of this space is taken the pair $(0,0)$, i. e. the set of elements $(a,a), a \in F$. From last relation (1) we get $(a,a) = (0,0)$. For any $x = (a,b)$, $-x = (b,a)$. It is clear that $x + (-x) = (a,b) + (b,a) = = (a+b, a+b) = (0,0)$.

The set of all pairs $(a,b) \in F \times F$ forms a structure of a linear space. Let

$x = (a_1,a_2) \in F \times F$, $y = (b_1,b_2) \in F \times F$.

Then

$a_i^{\alpha} = [L_{a_i}(\alpha), R_{a_i}(\alpha)]$, $b_i^{\alpha} = [L_{b_i}(\alpha), R_{b_i}(\alpha)], \alpha \in [0,1]$

For any $x, y \in F \times F$ define the scalar product as

$$\begin{aligned}
x \circ y = \frac{1}{2}\int_0^1 [(L_{a_1}(\alpha) - L_{a_2}(\alpha))(L_{b_1}(\alpha) - L_{b_2}(\alpha)) + \\
+ (R_{a_1}(\alpha) - R_{a_2}(\alpha))(R_{b_1}(\alpha) - R_{b_2}(\alpha))] d\alpha
\end{aligned} \tag{2}$$

It may be shown that this definition satisfies all requirements of the scalar product. We denote this space by LF. Norm in this space is defined as

$$\|x\|^2 = \frac{1}{2}\int_0^1 [(L_{a_1}(\alpha) - L_{a_2}(\alpha))^2 + (R_{a_1}(\alpha) - R_{a_2}(\alpha))^2] d\alpha \tag{3}$$

We define distance between two fuzzy numbers $a \in F$ and $b \in F$ as

$$\rho(a,b) = \|x - y\| \tag{4}$$

where $x = (a,0),\ \ y = (b,0)$.

3. Derivative of the fuzzy function

Now, let's consider fuzzy function $f(t) \in F$ for each $t \in [t_0, t_1]$ and define a derivative of the function $f(t)$.

For any $\alpha \in [0,1]$,

$$f_{\alpha}(t) = [L_{f(t)}(\alpha), R_{f(t)}(\alpha)], \alpha \in [0,1] \tag{5}$$

is called α -cut of the function $f(t)$.

Definition. Let there exists such $\varphi(t) \in F$, $\psi(t) \in F$, $t \in [t_0, t_1]$, that

$$\lim_{\Delta t \to 0} \frac{(f(t+\Delta t, 0) - (f(t), 0)}{\Delta t} = (\varphi(t), \psi(t)). \qquad (6)$$

Then the pair $(\varphi(t), \psi(t)) \in F \times F$ is called a derivative of the function $f(t)$ at the point $t \in [t_0, t_1]$. This definition may be written in the following form

$$\lim_{\Delta t \to 0} \frac{(f_\alpha(t+\Delta t, 0) - (f_\alpha(t), 0)}{\Delta t} = (\varphi_\alpha(t), \psi_\alpha(t)), \qquad (7)$$

where $(\varphi_\alpha(t), \psi_\alpha(t))$ are α -cut for the functions $\varphi(t)$, $\psi(t)$.

It is shown that, if $L_{f(t)}(\alpha), R_{f(t)}(\alpha)$ is continuous differentiable relatively t, then $f(t)$ is

differentiable. Each function $f(t)$ may be considered as an element $(f(t), 0)$ from $F \times F$. Then

$$(f_1(t) \pm f_2(t))' = f_1'(t) \pm f_2'(t). \qquad (8)$$

Now, let $f(t)$ be a pair of fuzzy functions, i.e.

$$f(t) = (f_1(t),\ f_2(t)), \forall t \in (t_0, t_1).$$

From relation

$$f(t) = (f_1(t), 0) + (0, f_2(t)) = (f_1(t), 0) - (f_2(t), 0)$$

we see, that the derivative of the function $f(t)$ also is a pair from $F \times F$.

For any $\eta = \eta(t) \in F \times F$, which $\eta'(t) \in F \times F$, consider the scalar product $f'(t) \circ \eta(t)$ defined by the formula (2). It can be shown that

$$\int_t^T f'(\tau) \circ \eta(\tau) d\tau = f(\tau) \circ \eta(\tau)\Big|_t^T - \int_t^T f(\tau) \circ \eta'(\tau) d\tau, \forall t, T \in (t_0, t_1).$$

(9)

One may show that this derivative satisfies the "necessary natural" conditions.

Example 1. Let $f(t)$ be fuzzy function whose α -cut is defined as follows:

$f_\alpha(t) = [t^2 - (1-\alpha)t, t^2 + (1-\alpha)t], t \geq 0.$ Then it is not difficult to show that

$$f_\alpha'(t) = (\varphi_\alpha(t), \psi_\alpha(t)),$$

where

$$\varphi_\alpha(t) = [2t - (1-\alpha), 2t + (1-\alpha)],\ \psi_\alpha(t) = 0.$$

Example 2. Let

$$f_\alpha(t) = [t^2 - \frac{1-\alpha}{t},\ t^2 + \frac{1-\alpha}{t}],\ t > 0.$$

In this case

$$\varphi_\alpha(t) = 0,\ \psi_\alpha(t) = [2t - \frac{1-\alpha}{t^2},\ 2t + \frac{1-\alpha}{t^2}].$$

Analogically we can define partial differential on $y_1, y_2, \ldots, y_n$.

Example 3. Let $a_1 \in F$, $a_2 \in F$ be fuzzy number and $u(y_1, y_2) = y_1^2 a_1 + y_2^2 a_2, y_i \in R$ be fuzzy function. It is no difficult to show that, for

$y_i \geq 0, i = 1,2,$

$$\frac{\partial u}{\partial y_i} = (2y_i a_i, 0).$$

In obverse

$$\frac{\partial u}{\partial y_i} = (0, 2|y_i| a_i, 0).$$

Also it is clear that

$$\frac{\partial^2 u}{\partial y_i^2} = (2a_i, 0).$$

4. Fuzzy elliptic equation

Let $D \in R^n$ be a given bounded domain with smooth boundary S and fuzzy function $u = u(y) \in F \times F$ depends on the parameter $y = (y_1, y_2, ..., y_n) \in D$, i.e. $u = u(y), \ y \in D$. We'll write $u \in C(D)$, if the function $\|u(y)\|$ continues on y in D. Analogically we can define $U \in C^1(D)$.

Consider the boundary problem

$$\Delta u = -f(y), \quad y \in D, \tag{10}$$

$$u(\xi) = g(\xi), \quad \xi \in S. \tag{11}$$

Let $f(y) = (f_1(y), f_2(y)) \in F \times F, y \in D,$ $g(\xi) = (g_1(\xi), g_2(\xi)) \in F \times F, \xi \in D.$

In the difference of traditional problems, here solution of the problem (6), (7) is fuzzy function $u = u(y) \in F$ or pair of the fuzzy function $u(y) = (u_1(y), u_2(y)) \in F \times F.$ For the of simplicity, this type functions we'll call fuzzy function. Equation (10) and boundary condition (11) we understand as equality pair of the domains.

Theorem 1. Let $f_i \in C^1(D) \cap C(\overline{D})$ and $g_i \in C(S), i = 1,2.$ Then there exists unequal solution $u(y) = (u_1(y), u_2(y)) \in F \times F, y \in D$ of the problem (10), (11).

It is interesting to investigate problem (10), (11), when $f(y), \ g(\xi)$ are fuzzy functions, e.i. $f(y) \in F, \ g(\xi) \in F, y \in D, \xi \in S.$ There is, that in this case solution of the problem (10), (11) also is fuzzy function from F .

Theorem 2. Let for any $y \in D$ and $\xi \in S$ be fuzzy function and $f \in C^1(D) \cap C(\overline{D}), \ g \in C(S).$ Then there exists unequal fuzzy function $u = u(y) \in F$ solution of the problem (10), (11).

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