WAVE PROPAGATION IN THREE-COMPONENT PHONONIC CRYSTALS
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Abstract: *the current work is devoted to mathematical modeling of wave propagation processes in inhomogeneous media. The waveguide and resonance properties of inhomogeneous one-dimensionally periodic media consisting of three components are considered within the framework of the one-dimensional approximation. To describe the propagation of acoustic waves in inhomogeneous one-dimensionally periodic structures, equations of stationary acoustic oscillations of pressure in a medium and boundary conditions (pressure and velocity continuity) are used.*
As a result of the study, a solution of the system for the fundamental cell was found. The transmission bands for waveguide modes and the dispersion relation for all waveguide modes are found. The dependence of the waveguide frequency on the linear concentrations of materials in the fundamental cell is found.
Keywords: *acoustic waves in inhomogeneous media, transmission bands, phonon crystal, waveguide modes, dispersion relation, vibration-insulating and sound-absorbing materials.*

РАСПРОСТРАНЕНИЕ ВОЛН В ТРЕХКОМПОНЕНТНЫХ ФОНОНИЧЕСКИХ КРИСТАЛЛАХ
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Аннотация: *данная работа посвящена математическому моделированию процессов распространения волн в неоднородных средах. В рамках одномерного приближения рассмотрены волноводные и резонансные свойства неоднородных одномерно-периодических сред, состоящих из трех компонентов. Для описания распространения акустических волн в неоднородных одномерно-периодических структурах используются уравнения стационарных акустических колебаний давления в среде и граничные условия (неразрывность давления и скорости). В результате исследования было найдено решение системы для фундаментальной ячейки. Найдены полосы пропускания волноводных мод и закон дисперсии для всех волноводных мод. Найдена зависимость частоты волновода от линейной концентрации материалов в основной ячейке.*
Ключевые слова: *акустические волны в неоднородных средах, полосы пропускания, фононный кристалл, волноводные моды, закон дисперсии, виброизоляционные и звукопоглощающие материалы.*

INTRODUCTION

The study of the problems of wave propagation in various media has been carried out since the 17th century. The analysis of these problems is an important tool for solving applied problems in inhomogeneous one-dimensionally periodic structures. Among the media in which acoustic waves propagate, the following frequently occurring structures can be distinguished: gas bubbles in a liquid, foam materials, inhomogeneous mixtures with periodically included components, various composite materials, granular and porous structures, etc. Similarly, we can consider examples with electromagnetic waves. The "quartz-water-air" structure studied in this paper is an example of a porous oil or waterflood formation. Important actions in applied problems are the determination of pass and stop bands, acoustic sounding - finding the values of linear concentrations from given values of phase velocities, as well as the study of the slowing down properties in inhomogeneous one-dimensionally periodic structures.

This work originates in the articles by S.V. Sukhinin in the field of studying the acoustics of inhomogeneous media. Works [1, 4] are taken as a basis. References [2, 3] contain the results of initial research and bibliography. In [4], the waveguide and resonance properties of inhomogeneous permeable one-dimensionally periodic structures consisting of two different media were studied using the one-dimensional approximation. The passbands and blockings are defined. A dispersion relation is obtained for all waveguide modes. Explicit expressions are found for low waveguide frequencies and the corresponding phase velocities of waveguide modes for mono- and polydisperse media. There are several studies showing that heterogeneous multi-component composites can have promising vibration-isolating and sound-absorbing properties (see, for example, [5]). Based on the results of works [6, 7], vibration isolation and the effectiveness of sound protection are illustrated by studying the propagation of a wave in a porous medium filled with liquid and gas.

The studies carried out in this work can be said to generalize the results of the above works. A feature of studying a three-component environment is an increase in computational difficulty, since as the number of boundaries between the media of the fundamental cell grows, so does the number of equations and the number of free parameters. As the goal of the current work, it is considered to investigate the properties of the three-component structure of inhomogeneities and solve the inverse problem of finding the concentration of components from the measured values of phase velocities. The results of the study can be used to obtain new composite materials, soundproof filters, improve damping devices and materials.

MODEL AND SYMMETRY PROPERTIES

An important feature of wave propagation in one-dimensional periodic structures is that the wave is the propagation of the oscillation phase in fundamental cells. Due to the large number of inhomogeneities, difficulties arise, which make it impossible to directly study the propagation of waves in inhomogeneous periodic structures. Therefore, the study of the fine structure of the frequency spectrum of the problem, which describes oscillations in one-dimensionally periodic structures, is of key importance.

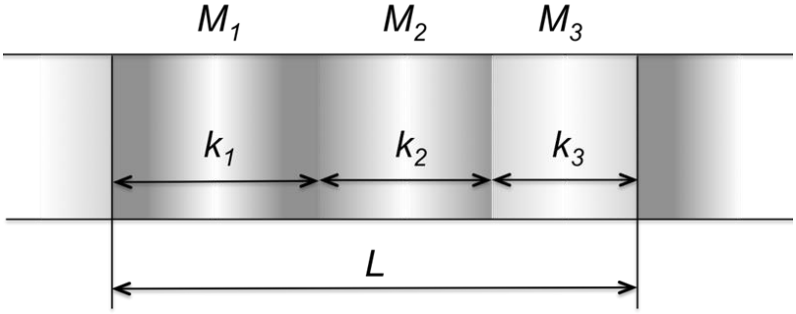

Fig. 1. Monodisperse chain of inhomogeneities

Let a heterogeneous one-dimensional periodic medium (Fig. 1) be composed of three media - $M_1 = \{ \mathrm{c}_1, \rho_1\}$, $M_2 = \{ \mathrm{c}_2, \rho_2\}$ и $M_3 = \{ \mathrm{c}_3, \rho_3\}$, where ρ_i and c_i are the density of the medium and the speed of sound, respectively, at rest. Let us assume that a chain of inhomogeneities, consisting of components M_1 and M_2, fills the component M_3 (composite matrix), and the linear concentration of the associated layers of components is equal to k_1, k_2 and k_3, respectively, and we assume that $k_1 + k_2 + k_3 = 1$. Indexes i = 1, 2, 3 correspond to the environments M_1, M_2, M_3, respectively. In what follows, we assume that the densities of the media satisfy the relation $\rho_1 > \rho_2 > \rho_3$. This chain is spatially periodic, in which the smallest spatial period is equal to L. We will consider a dimensionless variable in space $\hat{x} = x/L$, in further reasoning we will omit the lid for simplicity of description. In such notation, the smallest spatial period is equal to 1. A medium having such a length is called a fundamental cell. We also introduce auxiliary parameters $\tau_{ij} = \rho_i/\rho_j$, which will be an indicator of the ratio of densities for neighboring layers, and $\kappa_{ij} = c_i/c_j$ as the ratio of sound velocities in neighboring layers. It should be noted that the chain, which consists of three different media, has the property of spatial periodicity. Further calculations are carried out in dimensionless variables.

Stationary acoustic pressure oscillations with a circular frequency ω in the media M_1, M_2, M_3 are described by the following equations (Helmholtz):

$$p_{xx}^{(1)} + \Omega^2 p^{(1)} = 0,$$
$$p_{xx}^{(2)} + \Omega^2 \kappa_{12}^2 p^{(2)} = 0,\ p_{xx}^{(2)} + \Omega^2 \kappa_{13}^2 p^{(2)} = 0, \tag{1}$$

where $p^{(1)}$, $p^{(2)}$ and $p^{(3)}$are the acoustic pressures in the corresponding chain components, $\omega=2\pi f$ is the angular frequency, $\Omega=\omega L/c_1$ is the waveguide frequency (dimensionless oscillation frequency). The general form of the solution for each medium can be described by the following equations:

$$p^{(1)} = a_1 e^{i\Omega x} + b_1 e^{-i\Omega x},$$
$$p^{(2)} = a_2 e^{i\Omega\kappa_{12}x} + b_2 e^{-i\Omega\kappa_{12}x},\ p^{(3)} = a_3 e^{i\Omega\kappa_{13}x} + b_3 e^{-i\Omega\kappa_{13}x} \tag{2}$$

The conditions at the contact boundary of the components are continuity of pressure and velocity (of the normal component) – two types of boundary conditions.

$$\begin{aligned} p^{(1)} &= p^{(2)}, & \tau_{21} p_x^{(1)} &= p_x^{(2)}, \\ p^{(2)} &= p^{(3)}, & \tau_{32} p_x^{(2)} &= p_x^{(3)}, \\ p^{(3)} &= p^{(1)}, & \tau_{13} p_x^{(3)} &= p_x^{(1)}. \end{aligned} \tag{3}$$

Relations (1), (2), and (3) (hereinafter – problem T) completely describe the propagation of acoustic waves in inhomogeneous one-dimensionally periodic structures.

Symmetry properties. Due to the invariance of the wave equation with respect to local plane symmetries, the symmetry of problem T is identified by the symmetry of the sequence of inhomogeneities. By definition, all one-dimensional periodic structures admit the group {T}, hence the space of admissible solutions is decomposed into invariant subspaces [2]. Since a heterogeneous chain of inhomogeneities has the property of spatial periodicity, it can be attributed to phononic crystals (by definition). And as is known, invariants with respect to the group {T} with respect to the space variable are all crystallographic groups. Accordingly, by the property of commutativity {T}, we obtain that any representation of it is unitary and one-dimensional in the space of admissible solutions, so we can decompose it into invariant one-dimensional subspaces. The cardinality of the set of invariant resulting subspaces is infinite. Considering the symmetry properties of the medium, the solution of the problem satisfies the Floquet theorem (conditions for shifting the phase of oscillations):

$$p(x+1) = p(x)e^{i\xi}, \tag{4}$$

where i is the imaginary unit, ξ is the phase shift of oscillations in adjacent parts of the translation group, -π<ξ<π. In further reasoning, problem T together with (4) is called problem T(ξ).

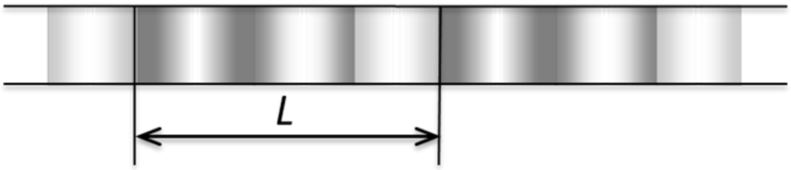

Fig. 2. Fundamental cell in a chain of inhomogeneities

Investigating the problem T(ξ) in one fundamental cell of the chain of translations, for example, on the interval 0<x<1 and continuing the solution from the unit interval to the whole line, we get the solution.

RESULTS

For each of the layers M_1, M_2, M_3, by assumption, the linear concentrations are equal to k_1, k_2, k_3, respectively. The most common and most common case is the monodispersed property of the chain of inhomogeneities. In the current problem, monodispersed is expressed in the condition $k_1 = k_2 = k_3 = \frac{1}{3}$.

Taking into account the general form of solutions (2), we obtain that in the fundamental cell the boundary conditions will be expressed by the following relations:

$$a_1 e^{i\Omega k_1} + b_1 e^{-i\Omega k_1} = a_2 e^{i\Omega k_1 \kappa_{12}} + b_2 e^{-i\Omega k \kappa_{12}}$$
$$a_2 e^{i\Omega(k_1+k_2)\kappa_{12}} + b_2 e^{-i\Omega(k_1+k_2)\kappa_{12}} = a_3 e^{i\Omega(k_1+k_2)\kappa_{13}} + b_3 e^{-i\Omega(k_1+k_2)\kappa_{13}}$$
$$\tau_{21}(a_1 e^{i\Omega k_1} - b_1 e^{-i\Omega k_1}) = \kappa_{12}(a_2 e^{i\Omega k_1 \kappa_{12}} + b_2 e^{-i\Omega k \kappa_{12}})$$
$$\tau_{32}\kappa_{12}\left(a_2 e^{i\Omega(k_1+k_2)\kappa_{12}} + b_2 e^{-i\Omega(k_1+k_2)\kappa_{12}}\right) = \kappa_{13}\left(a_3 e^{i\Omega(k_1+k_2)\kappa_{13}} + b_3 e^{-i\Omega(k_1+k_2)\kappa_{13}}\right)$$

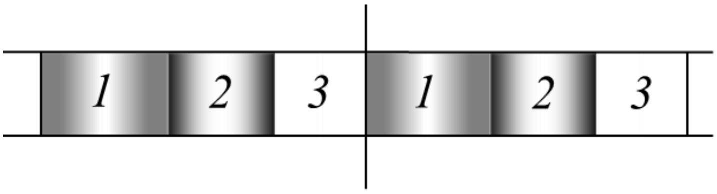

Fig. 3. Boundary of two fundamental cells

The kinematic and dynamic conditions at the boundary of two fundamental cells will slightly change their appearance due to the condition of the oscillation phase shift.

$$a_3 e^{i\Omega\kappa_{13}} + b_3 e^{-i\Omega\kappa_{13}} = a_1 e^{i\xi} + b_1 e^{-i\xi}$$
$$\tau_{13}\kappa_{13}(a_3 e^{i\Omega\kappa_{13}} - b_3 e^{-i\Omega\kappa_{13}}) = a_1 e^{i\xi} - b_1 e^{-i\xi}$$

In general, we can rewrite the properties for two neighboring cells like this:

$$\begin{gathered} p^{(3)}(1) = p^{(1)}(0)e^{i\xi}, \\ \tau_{13} p_x^{(3)}(1) = p_x^{(1)}(0)\mathrm{e}^{i\xi} \end{gathered} \tag{4}$$

Dispersion ratio. Combining the equations of the problem T(ξ) and equations (4) for the media M_1, M_2, M_3 into a system, we obtain the problem TM(ξ), which can be written as an equivalent SLE A(Ω)Y=0, where $Y = (a_1, b_1, a_2, b_2, a_3, b_3)$ is the vector of unknown constants in (2). The matrix A(Ω) of this system has the form:

$$\begin{pmatrix} e^{i\Omega k_1} & e^{-i\Omega k_1} & -e^{i\Omega\kappa_{12}k_1} & -e^{i\Omega\kappa_{21}k_1} & 0 & 0 \\ \tau_{21}e^{i\Omega k_1} & -\tau_{21}e^{-i\Omega k_1} & -\kappa_{12}e^{i\Omega\kappa_{12}k_1} & \kappa_{21}e^{-i\Omega\kappa_{12}k_1} & 0 & 0 \\ 0 & 0 & e^{i\Omega\kappa_{12}(k_1+k_2)} & e^{-i\Omega\kappa_{12}(k_1+k_2)} & -e^{i\Omega\kappa_{13}(k_1+k_2)} & -e^{-i\Omega\kappa_{13}(k_1+k_2)} \\ 0 & 0 & \tau_{32}\kappa_{12}e^{i\Omega\kappa_{12}(k_1+k_2)} & -\tau_{32}\kappa_{12}e^{-i\Omega\kappa_{12}(k_1+k_2)} & -\kappa_{13}e^{i\Omega\kappa_{13}(k_1+k_2)} & \kappa_{13}e^{-i\Omega\kappa_{13}(k_1+k_2)} \\ -e^{i\xi} & -e^{i\xi} & 0 & 0 & e^{i\Omega\kappa_{13}} & e^{-i\Omega\kappa_{13}} \\ -e^{i\xi} & e^{i\xi} & 0 & 0 & \tau_{13}\kappa_{13}e^{i\Omega\kappa_{13}} & -\tau_{13}\kappa_{13}e^{i\Omega\kappa_{13}} \end{pmatrix}$$

As is known, a nontrivial solution of a system of linear equations exists if the determinant of the matrix A(Ω) is equal to zero. Hence it turns out that the waveguide values of the TM problem are zeros of the analytical function detA(Ω). This means that the waveguide values $\Omega^*(\xi)$ of the TM problem on the real axis are discrete and depend on ξ continuously on the set $|\xi|<\pi$, and also on τ_{ij} and κ_{ij}.

For fixed $\tau_{ij}, \kappa_{ij}, k_i$, we obtain that the equation detA(Ω)=0 is nothing but the dispersion relation for all waveguide modes $\Omega^n = \Omega^n(\xi)$, $n = 1,2,\ldots$, which are connected components of the set of all waveguide values of the TM problem on the plane (ξ,Ω).

$$\begin{aligned}2\kappa_{12}\kappa_{13}(\cos(\xi) - \cos(\mathrm{k}_1\Omega)\cos(k_2\Omega\kappa_{12}\cos(k_3\Omega\kappa_{13})))\\ + (\kappa_{12}^2\tau_{32} + \kappa_{13}^2\tau_{23})\cos(\mathrm{k}_1\Omega)\sin(k_2\Omega\kappa_{12})\sin(k_3\Omega\kappa_{13})\\ + (\kappa_{13}^2\tau_{13} + \tau_{31})\kappa_{12}\sin(\mathrm{k}_1\Omega)\cos(k_2\Omega\kappa_{12})\sin(k_3\Omega\kappa_{13})\\ + (\kappa_{12}^2\tau_{12} + \tau_{21})\kappa_{13}\sin(\mathrm{k}_1\Omega)\sin(k_2\Omega\kappa_{12})\cos(k_3\Omega\kappa_{13}) = 0\end{aligned}$$

Long wave approach. An important study is the study of the propagation of low-frequency (long) waves along a one-dimensionally periodic chain of inhomogeneities. In this case, it turns out that the period of the structure and the size of the inhomogeneities are much smaller than the wavelength. As mentioned above, the waveguide values of the TM problem are solutions to the equation detA(Ω)=0. At Ω→0, the waveguide value is the waveguide value $\Omega^1$of the TM problem, which corresponds to the lowest frequency of waveguide oscillations of a monodisperse sequence of media. Next, let us introduce the definition for such an oscillation mode.

Assuming the decomposition of the determinant of the matrix, or the dispersion relation, in the creeping mode approximation problem, in a Taylor series at the point at Ω=0, and neglecting the terms of the order of three (Ω^3) and higher, we obtain an expansion for low-frequency waves:

$$2[\cos\xi - 1] + [k_1(k_1 + k_2\tau_{21} + k_3\tau_{31}) + k_2\kappa_{12}^2(k_1\tau_{12} + k_2 + k_3\tau_{32}) + k_3\kappa_{13}^2(k_1\tau_{13} + k_2\tau_{23} + k_3)]\Omega^2 = 0,$$

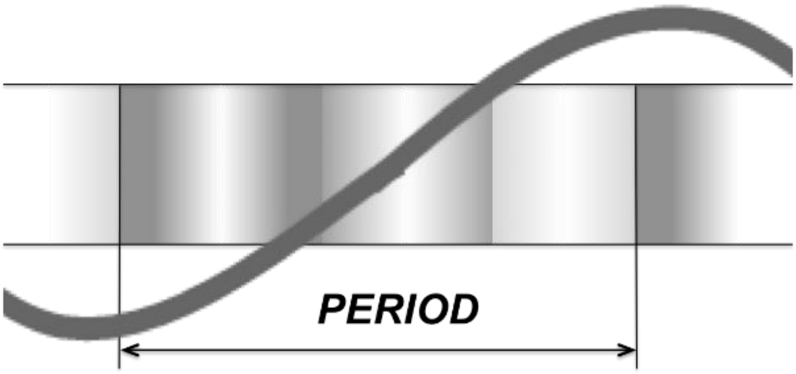

Fig. 4. Long (low frequency) wavewhich allows one to find an approximate expression for low wave frequencies of the creeping mode in a simple way:

$$\Omega^1(\xi) = \sqrt{\frac{2(1-\cos\xi)}{k_1(k_1 + k_2\tau_{21} + k_3\tau_{31}) + k_2\kappa_{12}^2(k_1\tau_{12} + k_2 + k_3\tau_{32}) + k_3\kappa_{13}^2(k_1\tau_{13} + k_2\tau_{23} + k_3)}}$$

Further, the calculations required numerical values of the densities and sound velocities of the media of the "quartz-water-air" structure:

Table 1. The speed of sound and the density of the medium

Name of the medium	Speed of sound in the medium, m/s	Density of the medium, g/cm^3
1. Quartz	6000	2.6
2. Water	1500	1.0
3. Air	330	1.2 10^{-3}

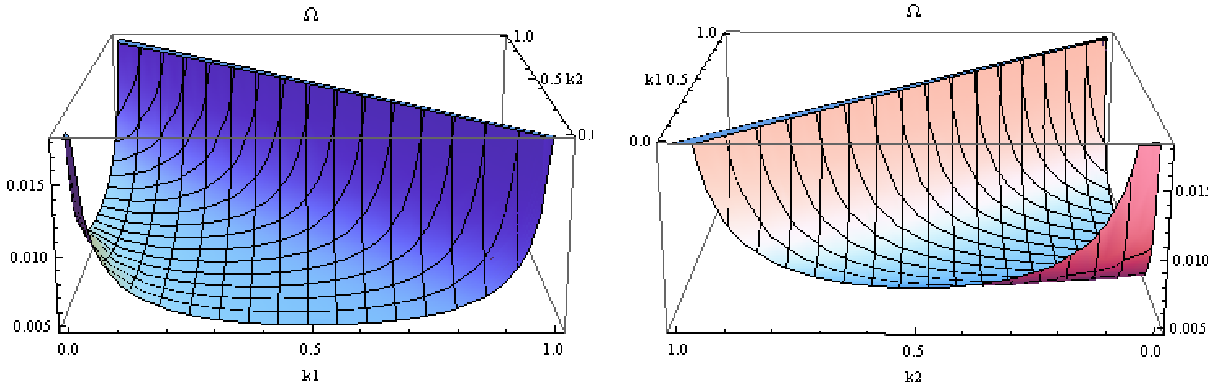

Fig. 5. Dependence of the waveguide frequency of the creeping mode on concentrations

It is also possible to visualize the results obtained more clearly - to consider the bandwidth in the context as a function of one variable.

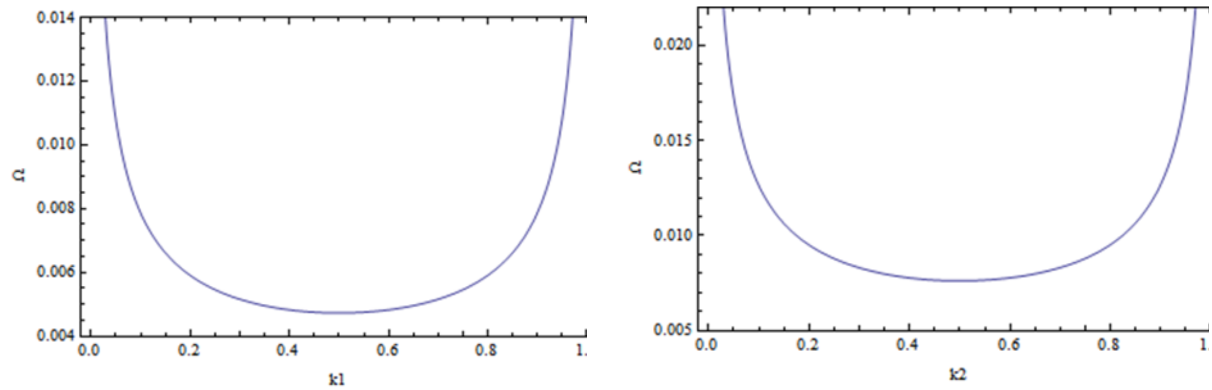

Fig. 6. Bandwidth, "quartz-air" $k_2 = 0$ (left),"water-air" $k_1 = 0$ (right)

It should be noted that the waveguide frequency of the creeping mode for Ω≈0 depends significantly on the linear concentrations. On Fig. 6 shows that the local minimum in both cases exists at the points k_1=0.5 for "quartz-air" and k_2=0.5 for "water-air", and at the extreme points 0 and 1 the creeping mode bandwidth expands.

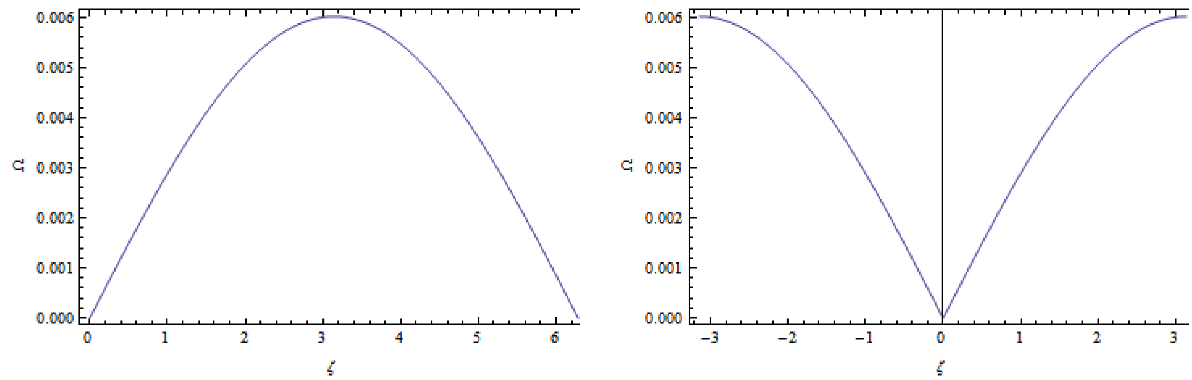

Fig. 7. Dependence of the waveguide frequency for the first waveguide mode on the wave ξ (left) and the same in the interval -π<ξ<π (right)

It can be seen from the nature of the dependence of the waveguide frequency on the phase shift parameter that the medium is highly dispersive.

CONCLUSION

Based on the results of this work, the following main conclusions can be drawn. In the general case, for a one-dimensional approximation, the propagation of acoustic waves in inhomogeneous one-dimensionally periodic structures is studied, and a dispersion relation is derived for all waveguide modes.

An expression for low waveguide frequencies is explicitly obtained in the long-wave approximation. The pass and stop bands for the selected chain of inhomogeneities "quartz-water-air" are determined.

The results of the work can be used for further study of wave propagation in layered phononic crystals.

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